UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 1st FEBRUARY 2001

Organised by the **United Kingdom Mathematics Trust from the School of Mathematics, University of Leeds**

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. **A** The differences are, respectively, 11, 8, 10, 9 and 9.

2. **E** Over a two-week period, the required number of visitors is $14 \times 30\,000 = 420\,000$. Hence the number required this week is $420\,000 - 120\,000 = 300\,000$.

3. **A** Since $\frac{1}{4} = \frac{4}{16}$ and $\frac{1}{8} = \frac{2}{16}$, the number midway between them is $\frac{3}{16}$.

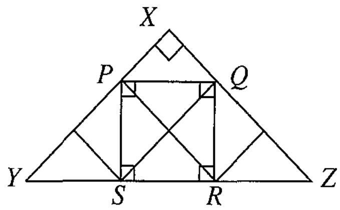
4. **E** Martha has 5 children, 20 grandchildren and 60 great-grandchildren i.e. 85 descendants.

5. **C** Angle $PSR = 41°$ (opposite angles of a parallelogram are equal). Therefore $x = 41 + 83$ because the exterior angle of a triangle is equal to the sum of the two interior opposite angles.

6. **D** $105 = 3 \times 5 \times 7$.

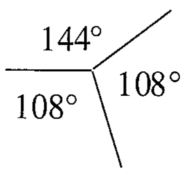
7. **C** The thickness of one sheet = $(54 \div 500)$ mm = $(108 \div 1000)$ mm = 0.108 mm = 0.1 mm to one significant figure.

8. **A** The diagram shows that triangle XYZ may be divided into 9 congruent triangles. The square $PQRS$ is made up of 4 of these 9 triangles.



9. **D** The number of complete days which have elapsed is $366 + 31 \approx 400$.
Hence the number of seconds $\approx 400 \times 24 \times 60 \times 60 \approx 10000 \times 3600 = 3{\cdot}6 \times 10^7$.

10. **B** The interior angle of a regular pentagon is 108°. Therefore each interior angle of the regular polygon formed by the inner sides of the pentagons is $(360 - 2 \times 108)° = 144°$. The exterior angle of this regular polygon is 36° and hence it has $360 \div 36$, i.e. 10, sides. Therefore 7 more pentagons are required.



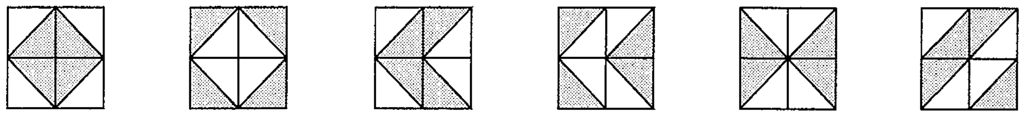
11. **B** Imagine that the card is transparent. Then, after the first rotation the shape on the card will look like . The diagonal line now runs North-East to South West and so it will appear the same after the second rotation. The line which runs North-South on the left of the figure will run East-West at the bottom of the figure after this second rotation, while the line which runs North-South on the right of the figure will run East-West at the top of the figure. Thus the final appearance will be .

12. **E** Every long-sleeve shirt requires 2 cuff buttons; hence for every one long-sleeve shirt produced by the factory, a total of 20 front buttons are used. The long-sleeve shirt requires 8 of these and therefore the remaining 12 will be the front buttons on 2 short-sleeve shirts. Thus the required ratio is 1:2.

13. **B** The general form of the example given is $1 + 3 + 5 + \ldots + (2n - 1) + \ldots + 5 + 3 + 1 = (n - 1)^2 + n^2$. Therefore, for $n = 1001$:

$$1 + 3 + 5 + \ldots + 1999 + 2001 + 1999 + \ldots + 5 + 3 + 1 = 1000^2 + 1001^2.$$

14. **D** The only possible distinct patterns are:



15. **C** $3\sqrt{11} = \sqrt{9} \times \sqrt{11} = \sqrt{9 \times 11} = \sqrt{99}$; similarly, $4\sqrt{7} = \sqrt{112}$; $5\sqrt{5} = \sqrt{125}$; $6\sqrt{3} = \sqrt{108}$; $7\sqrt{2} = \sqrt{98}$. Hence $4\sqrt{7}$, $5\sqrt{5}$ and $6\sqrt{3}$ are all greater than 10.

16. **B** One cake and one bun cost a total of 62p. Note that $512 \div 62 = 8$ remainder 16 and note also that $16 = 39 - 23$. Hence $512 = 8(39 + 23) + 16 = 8(39 + 23) + 39 - 23 = 9 \times 39 + 7 \times 23$. As 39 and 23 do not have a common factor, other than 1, Helen must have bought 9 cakes and 7 buns.

17. **C** In each minute, the two clocks showed the same time for 40 seconds i.e. they showed the same time for $\frac{40}{60}$ of the day.

18. **E** Let the athlete take x minutes to cycle one mile. Then he takes $\frac{3x}{2}$ minutes to run one mile, and $3x$ minutes to walk one mile. Therefore: $3x + \frac{3x}{2} + x = 3x + 10$ i.e. $x = 4$.
The cyclist takes 12 minutes to walk the first mile, 6 minutes to run the second mile and 4 minutes to cycle the third mile: a total time of 22 minutes.

19. **E** The ratio length:breadth of the smaller rectangles is 5:4. Let the length and breadth of these rectangles be $5x$ cm and $4x$ cm respectively. The area of the large rectangle, in cm^2, is $9 \times 20x^2 = 180x^2$ and the only one of the alternatives which is a product of 180 and a perfect square is 1620, which corresponds to $x = 3$.

20. **D** Note that the sum of the whole numbers from 1 to 9 inclusive is 45, a multiple of 3. Thus the stated problem may be reduced to finding the number of ways of choosing two of these numbers whose sum is a multiple of 3. There are 12 ways of doing this: 1, 2; 1, 5; 1, 8; 2, 4; 2, 7; 3, 6; 3, 9; 4, 5; 4, 8; 5, 7; 6, 9; 7, 8.

21. **A** Given that $0 < x < 1$, we may deduce that $x^2 + x > x^2 > x^3 > x^4$ and also that $x^2 + x > x^2 + x^3$.

22. **C** For the triangular faces of the resulting solid to be equilateral, it is necessary for each of the solids removed at the corners to be a regular tetrahedron. Removing a regular tetrahedron at a corner in this manner does not change the total length of the edges of the solid as the perimeter of the equilateral triangle created is equal to the sum of the other three edges of the removed tetrahedron. The original tetrahedron had six edges, all of side 6 cm, and therefore the total length of the edges of the resulting solid is 36 cm.

23. **B** Note that $\frac{n+3}{n-1} = \frac{n-1}{n-1} + \frac{4}{n-1} = 1 + \frac{4}{n-1}$. Thus $\frac{n+3}{n-1}$ is an integer if and only if $n - 1$ divides exactly into 4 (or -4). The values of n for which this is true are $-3, -1, 0, 2, 3$ and 5.

24. **D** The totals of the top row and the completed main diagonal are 30 and 39 respectively and therefore the ten consecutive numbers in question must be those from 30 to 39 inclusive. The number in the bottom right-hand corner must be one of 1, 2, 8, 15 or 16. Taking 1 or 2 makes the diagonal (from top left to bottom right) add up to less than 30, while taking 15 or 16 produces a total greater than 39. Hence 8 must go in the bottom right-hand corner. It now follows that * must be replaced by 15, since if 15 is placed in one of the other three vacant squares, we get a total of 45 (second column, too big), 34 (third column, same as diagonal) or 47 (third row, too big). You should check that the square can now be completed successfully.

25. **D** Let O be the centre of the circle and let the points where the arcs meet be C and D respectively. $ABCD$ is a square since its sides are all equal to the radius of the arc CD and $\angle ACB = 90°$ (angle in a semicircle). In triangle OCB, $CB^2 = OC^2 + OB^2$; hence $CB = \sqrt{2}$ cm. The area of the segment bounded by arc CD and diameter CD is equal to the area of sector BCD – the area of triangle BCD, i.e.
$\left(\frac{1}{4}\pi\left(\sqrt{2}\right)^2 - \frac{1}{2} \times \sqrt{2} \times \sqrt{2}\right)$ cm^2 i.e. $\left(\frac{1}{2}\pi - 1\right)$ cm^2.
The unshaded area in the original figure is, therefore, $(\pi - 2)$ cm^2. Now the area of the circle is π cm^2 and hence the shaded area is 2 cm^2.
(Note that the shaded area is equal to the area of square ABCD. This can be proved by showing that the areas of the two regions shaded in the lower diagram are equal. This is left as a task for the reader.)